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IV.—*On a new Barometric Formula for Mountain Heights, in which the Hygrometric Condition of the Atmosphere is systematically considered.* *By* HENRY LAUS RENNY, *M. R. I. A., Lieutenant, Royal Engineers, Retired List.*

Read January 8th, 1855.

IN the most approved formulæ employed in determining the height of mountains by means of the barometer, we find attempts to make allowance for the hygrometric state of the atmosphere, which attempts are as loose as they are inaccurate. It is therefore desirable to reconsider systematically the entire question, in order to obtain a formula free from the defects of those at present in use, the necessity of which cannot be doubtful to any one who is aware that in the formula given by LA PLACE and POISSON, the mode of making allowance for the hygrometric state of the atmosphere increases, instead of diminishing the error whenever the temperature of the atmosphere is below the freezing point; which circumstance has already been brought before the notice of the Academy in a paper submitted to it by Dr. APJOHN, and published in the "Proceedings of the Royal Irish Academy," vol. II. p. 563.

1. The ordinary method of obtaining the fundamental equation for determining the height by means of the barometer, viz. $h = c \log \frac{p}{p'}$, in which h indicates the height of one station above the other, c a constant quantity, and p, p' the pressures of the atmosphere, which method consists in regarding the heights of the stations to vary as the logarithms of the pressures—is mathematically correct, and sufficient for practical purposes, so long as the question has reference to only one elastic gas, such as dry air; but in a question which has reference to a union or mixture of two elastic gases, such as dry air and vapour of water, having different elastic forces, it is necessary to employ the integral calculus; and the following calculations are made after the method of POISSON, which

may be found in POISSON's "Mecanique" (Paris edition, 1811, vol. II. book iv. chap. v. p. 429), with such additions and modifications as the more correct consideration of the subject renders necessary. The method employed by POISSON is substantially the same as that employed by LA PLACE, which may be found in the "Mecanique Celeste."

2. Let—

π be the elastic force of a given mixture or union of dry air and vapour of water, being measured by the pressure to which the said mixture is exposed.

τ be the temperature of said mixture, measured by Fahrenheit's thermometer.

F be elastic force or tension of vapour of water of said mixture.

ρ be specific gravity or density of said mixture.

A be ratio of elasticity to density (also called modulus of elasticity) of said mixture, and equal to $\frac{\pi}{\rho}$.

a be modulus of elasticity of dry air at freezing point.

z be height above the level of the sea of an extremely thin stratum of a mixture of dry air and vapour of water, having a temperature τ, and under a pressure π, the elastic force of the vapour of water being F.

g' be force of gravity at distance z above the level of the sea.

g be force of gravity at the level of the sea.

ψ be latitude of the stations of barometric observations.

ζ be half the increase of gravity from the equator to the pole of the earth $=$ 0·002695.

r be the radius of the earth $=$ 20898240 feet.

M be modulus of common logarithms.

l be expansion of gas for each degree of Fahrenheit $=$ 0·002084.

m be expansion of quicksilver for each degree of Fahrenheit $=$ 0·0001.

Let, moreover,—

h, h' be heights above the level of the sea of the lower and upper stations of barometric observations.

p, p' be pressures of the atmosphere at lower and upper stations.

f, f' be tensions or forces of aqueous vapour at lower and upper stations.

β, β' be barometric observations at lower and upper stations.

T, T' be temperatures of the quicksilver of the barometers at lower and upper stations, as shown by the attached thermometer.
t, t' be temperatures of the atmosphere at lower and upper stations, as shown by the detached thermometer.
$H = h' - h$ be the required height of the upper station above the lower one.

Now, according to our definitions, we have

$$A = \frac{\pi}{\rho}, \tag{1}$$

and whereas the mixture or union of dry air and vapour of water is exposed to a pressure π, and that the vapour of water sustain a part of the pressure $= F$, it is obvious that the dry air of the mixture is exposed to a pressure $= \pi - F$; it is, moreover, well known that the densities of dry air and vapour of water, under the same or equal pressure, are to each other as 8 to 5; also that, *cæteris paribus*, the densities of gas vary as the pressures to which it is exposed. If, therefore, we assume unity to represent the density of the dry air of the mixture, we have $\frac{5}{8} . \frac{F}{\pi - F}$, to represent the density of the vapour of water of said mixture, also $1 + \frac{5}{8} . \frac{F}{\pi - F} = \frac{\pi - \frac{3}{8}F}{\pi - F}$ to represent the density of the mixture itself; but, according to our definitions, this last density is ρ; therefore, as $\frac{\pi - \frac{3}{8}F}{\pi - F} : 1 :: \rho :$ the density of the dry air of the mixture $= \rho . \frac{\pi - F}{\pi - \frac{3}{8}F}$. But the said dry air is under a pressure $= \pi - F$, and as the densities of gas vary *cæteris paribus* as the pressures to which it is exposed, we have as an expression for the density of the dry air of the mixture under a pressure π the quantity $\rho . \frac{\pi}{\pi - \frac{3}{8}F}$, at the same time that the quantity ρ indicates the density of the mixture itself under an equal pressure π, and of the same temperature T. The moduli of elasticity (*vide* Definitions) of different gases, under equal pressures, vary obviously as the reciprocals of their densities; therefore—

$$\text{As } \frac{1}{\rho} : \frac{1}{\rho \frac{\pi}{\pi - \frac{3}{8}F}} :: A : A . \frac{\pi - \frac{3}{8}F}{\pi}.$$

But according to our definitions, the modulus of elasticity of dry air at the freezing point is a, and, making allowance for increase of elasticity in consequence of increase of temperature, we have $a\,.\,\{1+l(T-32)\}$ for the expression of the modulus of elasticity of the dry air of the mixture having a temperature T. Therefore, equating these two expressions, we have

$$A\,\frac{\pi-\frac{3}{8}F}{\pi}=a\,.\,\{1+l(T-32)\} \text{ or } A=a.\{1+l(T-32)\}.\frac{\pi}{\pi-\frac{3}{8}F}.$$

Substituting this value of A in equation (1) we have

$$\rho=\frac{1}{a\,.\,\{1+l(T-32)\}}\cdot\{\pi-\tfrac{3}{8}F\}; \tag{2}$$

in which equation ρ indicates the density or specific gravity of mixture or union of dry air and vapour of water, having a temperature T, and under a pressure π; l, the expansion of gas for each degree of Fahrenheit; and a, the modulus of elasticity of dry air at the freezing point; and F, the tension or elastic force of the vapour of water of said mixture, having the temperature T.

3. Let us now suppose a cylindrical column of a mixture or union of dry air and vapour of water to exist between the stations of barometric observations, and let us suppose L to express the area or surface of an extremely thin stratum of said mixture at a distance above the level of the sea $=z$, then dz is the thickness of such stratum, $+Ldz$ is the solid content of such stratum, and $-Ldz\rho g'$ is the weight of said stratum. Moreover, $L\pi$ and $L(\pi+d\pi)$ express the pressures on the upper and lower surfaces of the stratum, the difference of which pressures is obviously equal to the weight of the stratum: we have, therefore, $L(\pi+d\pi)-L\pi=-Ldz\rho g'$, or $d\pi=-dz\rho g'$. Substituting in this last equation the value of ρ given by equation (2), we have

$$d\pi=\frac{-g'dz}{a\{1+l(T-32)\}}\cdot(\pi-\tfrac{3}{8}F).$$

But $g'=g\,.\,\frac{r^2}{(r+z)^2}$; therefore we have, after substitution and modification,

$$\frac{d\pi}{\pi-\frac{3}{8}F}=\frac{-gr^2}{a\{1+l(T-32)\}}\cdot\frac{dz}{(r+z)^2}. \tag{3}$$

Comparing equation (3) with corresponding equation of Poisson's "Mecanique" (vol. II. chap. v. article 542, Paris edition, 1811), viz.

$$\frac{dp}{p} = \frac{-gr^2}{a(1+\alpha x)} \cdot \frac{dz}{(r+z)^2},$$

and recollecting that, in Poisson's equation, the product αx has the same signification as $l(T-32)$ of our equation, x being the number of centigrade degrees above freezing point, and α being the expansion of gas for each centigrade degree: a being, in Poisson's equation, as in ours, the modulus of elasticity of dry air (the only air considered by Poisson); also p having the same signification as π, and the other characters of Poisson's equation being the same as in ours. It appears that, in Poisson's equation, the denominator p of the left-hand side of the equation is too great; by way of compensation, Poisson, following the recommendation of La Place, increases the denominator of the right-hand side of the equation, by increasing the quantity α,—making it 0·004, instead of 0·00375, its real value. Now, though this increase of the value of α may improve Poisson's equation, when the temperature is above the freezing point, it unquestionably increases the error when the temperature is below the freezing point, because, under such circumstances, x in Poisson's equation becomes subtractive, and the denominator of the right-hand side of Poisson's equation, instead of being increased, becomes diminished. Moreover, the substitution (even when the temperature is above the freezing point) of 0·004 instead of 0·00375 is, at best, but a guess as to quantity, and, like all guesses, more or less uncertain: hence the necessity of some effort to improve the formulæ given by La Place and Poisson.

4. Let us now proceed to integration. In order to do so, we ought to be acquainted with the law of variation of temperature of the atmosphere between the stations of barometric observations; also the law of variation of the elastic forces or tensions of vapour of water between the said stations. In the absence of correct information relative to the first law, it is assumed that the temperature of the atmosphere between the stations varies uniformly; a mere supposition (sometimes true, sometimes far from it). According to such assumption, we have to replace the variable $(T-32)$, by the arithmetic mean of the temperatures of the stations as ascertained by the detached thermometers, that is, to replace

$$(T-32) \text{ by } \frac{(t-32)+(t'-32)}{2}=\frac{\overline{t+t'}-64}{2}.$$

With respect to the laws of variation of forces of aqueous vapour, it appears, on consulting the Table of Forces of Aqueous Vapour in TURNER's "Chemistry" (seventh edition, pp. 1248 and 1249), that such forces form *quam proxime* a geometric series, when the temperatures form an arithmetic one. As, therefore, we assume the arithmetic mean $\frac{t+t'-64}{2}$ to replace the variable $(T-32)$, we can do nothing better to replace our variable F than to assume, as its representative, the geometric mean of forces of vapour, viz. $\sqrt{(ff')}$.

Making such substitutions in equation (3), we have

$$\frac{d\pi}{\pi-\frac{3}{8}\sqrt{(ff')}}=\frac{-gr^2}{a\left\{1+l\frac{t+t'-64}{2}\right\}}\cdot\frac{dz}{(r+z)^2}. \tag{4}$$

Integrating equation (4), we have

$$\text{hyp. log }(\pi-\tfrac{3}{8}\sqrt{(ff')}=C+\frac{g}{a\left\{1+l\frac{t+t'-64}{2}\right\}}\cdot\frac{r^2}{(r+z)}.$$

Replacing successively in this last equation the quantity π, by its values at the lower and upper stations, viz., p and p'; z by its corresponding values h and $(h+H)$; substituting for hyperbolic logarithms common ones, and subtracting one equation from the other, we have, after suitable modifications,

$$H=\frac{a.\left(1+l.\frac{t+t'-64}{2}\right)}{Mg}\cdot\left\{1+\frac{2h+H}{r}+\frac{h(h+H)}{r^2}\right\}.\text{com. log}\frac{p-\frac{3}{8}\sqrt{(ff')}}{p'-\frac{3}{8}\sqrt{(ff')}}.$$

Omitting in this last equation the quantity $\frac{h(h+H)}{r^2}$ as too small to be usefully retained, and assuming K_1 to represent the quantity $\frac{a}{Mg}$, we have

$$H=K_1\left\{1+l\frac{t+t'-64}{2}\right\}.\left\{1+\frac{2h+H}{r}\right\}.\text{com. log}\frac{p-\frac{3}{8}\sqrt{(ff')}}{p'-\frac{3}{8}\sqrt{(ff')}}.$$

We have now to replace p and p' by their values expressed in terms of the barometric observations, viz. β and β'. In so doing, we must make allowance for the difference of the specific gravities of the quicksilver of the barometers at the stations of observation ; at the lower station the specific gravity of the quicksilver is rendered greater than that at the upper station, by the greater power of gravity at the lower station. But the specific gravity of the quicksilver at the upper station is rendered greater than that at the lower station by diminution of temperature. Considering these two totally distinct causes of difference of specific gravities, if we assume to represent p of the lower station, the quantity $\beta . \frac{(r + h')^2}{(r + h)^2}$ we have to represent p' of the upper station, the quantity

$$\beta' . \frac{1 + m(T - 32)}{1 + m(T' - 32)}.$$

But $\frac{(r+h)^2}{(r+h)^2} = \frac{(\overline{r+h} + \overline{h'-h})^3}{(r+h)^2} = \frac{(\overline{r+h} + H)^2}{(r+h)^2} = \left(1 + \frac{H}{r+h}\right)^2 =$ (by omitting quantities too small to be usefully retained) $1 + \frac{2H}{r}$.

Moreover, $\frac{1 + m.(T - 32)}{1 + m.(T' - 32)} = \{1 + m\,(T - 32)\} . \{1 + m . (T' - 32)\}^{-1}$ $= \{1 + m\,(T - 32)\}.\{1 - m\,(T' - 32) + Am^2 + Bm^2 + Cm^3 + \&c.\} =$ (by omitting second and higher powers of m, as too small to be usefully retained) $1 + m\,(T - T')$. Thus we have to replace p and p', the expressions

$$\beta\left(1 + \frac{2H}{r}\right) \text{ and } \beta'.\{1 + m(T - T')\}.$$

Substituting these, we have

$$H = K_1\left\{1 + l . \frac{t + t' - 64}{2}\right\} . \left\{1 + \frac{2h + H}{r}\right\} . \text{com. log} \frac{\beta\left(1 + \frac{2H}{r}\right) - \frac{3}{8}\sqrt{(ff')}}{\beta'\{1 + m(T - T')\} - \frac{3}{8}\sqrt{(ff')}}.$$

K_1 is a function of g, and as g varies with the latitude of the stations of barometric observations, it is necessary to consider K_1 as a function of the latitude, and to seek its value as such. For this purpose, let K correspond to latitude 45°, and let G be corresponding force of gravity, while K_1 corresponds to latitude ψ, of which g is the corresponding force of gravity (*vide* Definitions). Then, according to POISSON'S "Mecanique" (Paris edition, 1811, vol. I. book ii. article 194), we have

$$g = G(1 - \zeta \cos 2\psi).$$

But $K_1 = \frac{a}{Mg} = \frac{a}{M\,.\,G(1 - \zeta \cos 2\psi)} = \frac{a}{MG}(1 - \zeta \cos 2\psi)^{-1}$ = (by omitting second and higher powers of ζ, as too small to be usefully retained) $\frac{a}{MG}(1 + \zeta \cos 2\psi)$. But $K = \frac{a}{MG}$, therefore $K_1 = K\,.\,(1 + \zeta \cos 2\psi)$.

Substituting this value of K_1 in our last equation, we have

$$H = K\,.\,\{1 + \zeta \cos 2\psi\}\,.\left\{1 + \frac{2h + H}{r}\right\}.\left\{1 + h\frac{t + t' - 64}{2}\right\}$$

$$.\ \text{com. log}\ \frac{\beta\left(1 + \frac{2H}{r}\right) - \frac{3}{8}\sqrt{(ff')}}{\beta'\{1 + m(T - T')\} - \frac{3}{8}\sqrt{(ff')}}$$

In this last equation K is a constant quantity corresponding to latitude 45°.

Let us now compare this last equation with the similar one of POISSON'S "Mecanique" (Paris edition, 1811, vol. II. book iv. chapter v.), viz.—

$$z = 18336\,.\,\{1 + 0{\cdot}00287 \cos 2\psi\}\,.\left\{1 + \frac{2(t + t')}{1000}\right\}.\left\{1 + \frac{z}{r}\right\}.\left\{\log\frac{h}{h'} + 2\log\left(1 + \frac{z}{r}\right)\right\},$$

and recollecting that in POISSON'S equation h and h' (which have the same signification as β and β' of our equation) are supposed to be corrected for difference of temperatures, as shown by the attached thermometers; consequently $\log\frac{h}{h'} + 2\log\left(1 + \frac{z}{r}\right) = \log\frac{h\left(1 + \frac{z}{r}\right)^2}{h'}$ = (by omitting quantities too small to be usefully retained) $\log\frac{h\left(1 + \frac{2z}{r}\right)}{h'}$ has the same signification as

$$\log\frac{\beta\left(1 + \frac{2H}{r}\right)}{\beta'\{1 + m\,.\,T - T')\}}$$

of our equation; recollecting also, that in POISSON'S equation the lower station is supposed to be at the level of the sea (which supposition injures POISSON's formula by diminishing its generality), it appears that in systematically taking into consideration the hygrometric state of the atmosphere, our formula differs from that of POISSON only in subtracting the quantity $\frac{3}{8}\sqrt{(ff')}$ from the barometric observations properly corrected. The constant coefficient, 18336 metres, of POISSON's equation,—which is equal to 60158·6 English feet, having been calculated for latitude 45°, without a systematic consideration of the hygrometric state of the atmosphere,—ought to be replaced by a constant obtained from a considerable number of accurate barometric observations, taken in connexion with accurate observations of the wet and dry thermometers.

5. Let us now consider if we can simplify our formula. POISSON says that the corrections dependent on the height of the stations above the level of the sea, and on the height of one station above the other, may be omitted in calculations not requiring any very great nicety, if we increase the constant; and this opinion of POISSON, as well as the pernicious advice given by him, as also by LA PLACE, with a view to allow for the vapour of water of the atmosphere, are adopted in BAYLEY's Astronomical Tables. Notwithstanding such authority, this advice ought not to be followed, as it sometimes leads to errors greater than the unavoidable errors of observation. In fact, these corrections enter the formula in two ways altogether different. In the expression $\left(1+\frac{2h+H}{r}\right)$, which is a multiplier of the constant, the correction arises from the height of the stations above the level of the sea, and in the expression $\beta\left(1+\frac{2H}{r}\right)$ the correction arises from the height of one station above the other. As the calculations are not very troublesome, it is desirable, in all nice and accurate barometric observations, to retain these corrections;—to the discretion of the calculator must be left when to retain and when to omit them.

It is now advisable to make some remarks touching the elastic forces of vapour of water found in our formula. The quantities f and f' (according to our definitions) represent the actual forces or tensions of the vapour of water of the atmosphere of the lower and upper stations. These forces differ from the elastic forces of the dew-points by quantities so extremely small that we shall employ, instead of them, the forces of the dew-points. Hygrometric observations should always, if possible, be made in connexion with barometric ones, but when this cannot be done, it is better, I believe, to obtain from an approved table of forces of aqueous vapour approximate values of f and f', such as belong to temperatures 3° or 4° below the temperatures t and t', than to neglect altogether the hygrometric state of the atmosphere. With respect to the quantity $K(1+\zeta\cos 2\psi)$, we may substitute for it a constant calculated for a particular latitude, which will answer well enough for latitudes no more than one degree north or south. Such constant may be designated a Local Constant, and the formula containing it a Local Formula.

Let us now reproduce our formula—

$$H = K.\{1+\zeta\cos 2\psi\}.\left\{1+\frac{2h+H}{r}\right\}.\left\{1+l.\frac{t+t'-64}{2}\right\}$$

$$.\,\text{com. log}\,\frac{\beta\left(1+\frac{2H}{r}\right)-\frac{3}{8}\sqrt{(ff')}}{\beta'\{1+m(T-T')\}-\frac{3}{8}\sqrt{(ff')}}. \qquad (A)$$

Let us replace $K(1+\zeta\cos 2\psi)$ by a constant calculated for a particular latitude. Let such constant be N, then

$$H = N.\left\{1+\frac{2h+H}{r}\right\}.\left\{1+l.\frac{t+t'-64}{2}\right\}$$

$$.\,\text{com. log}\,\frac{\beta\left(1+\frac{2H}{r}\right)-\frac{3}{8}\sqrt{(ff')}}{\beta'\{1+m(T-T')\}-\frac{3}{8}\sqrt{(ff')}}. \qquad (B)$$

Let us now omit the quantities $\left(1+\frac{2h+H}{r}\right)$ and $\left(1+\frac{2H}{r}\right)$; when such may be done without serious error, we have

$$H = N.\left\{1+l.\frac{t+t'-64}{2}\right\}.\,\text{com. log}\,\frac{\beta-\frac{3}{8}\sqrt{(ff')}}{\beta'\{1+m(T-T')\}-\frac{3}{8}\sqrt{(ff')}}. \qquad (C)$$

K is the constant belonging to latitude 45° $\zeta = 0{\cdot}002695$; $r = 20898240$ feet ; $l = 0{\cdot}002084$; $m = 0{\cdot}0001$. The term $\sqrt{(ff')}$, which is the geometric mean of forces of aqueous vapour, is readily obtained from an approved table of forces of aqueous vapour.

The formula (A) is the formula in its most general form, the constant K being calculated for 45° latitude. The formula (B) is a local formula. The formula (C) is also a local formula, and may be employed in England when the upper station is not more than 500 feet above the lower one, or more than 1000 feet above the level of the sea, and when the barometric observations are of the rough kind. I strongly recommend, however, the use of formula (B) in *all* circumstances.

A new constant is obviously a desideratum in all these formulæ.

Before this paper is brought to a conclusion, some remarks in reference to papers on this subject, and published in the second and third volumes of the "Proceedings of the Royal Irish Academy," may not be inappropriate.

In these papers it is assumed, that because the thickness of a portion of dry air, by becoming united to a portion of vapour of water having a tension F, both under a pressure π, is to the thickness of dry air, previous to its union, under an equal pressure, as $\pi : \pi - F$; so that dry air necessarily receives an increase of thickness by becoming united with vapour of water, the pressure continuing unchanged. It follows that in employing barometric observations for ascertaining the height of mountains, when we take into consideration the hygrometric state of the atmosphere, we ought to allow a corresponding increase of height in the proportion, $\pi - F : \pi$, calculated by summation of variable quantities by means of series, or of the integral calculus.

Now if the increase of height, in consequence of the atmosphere containing aqueous vapour, were analogous to the increase of height produced by increase of temperature, which causes an increase of the elastic force of the atmosphere, without *any* addition of material substance, the method pursued in former papers would be substantially correct. But vapour of water, by becoming united to dry air, adds its own weight to that of the dry air. Thus the analogy fails altogether; so much so indeed, that if the elastic forces of dry air and of vapour of water were equal, the formula for dry air alone would be the same as that for dry air united to vapour of water, notwithstanding that dry air necessarily expands by addition of vapour of water, the pressure continuing unchanged. But the elastic forces of vapour of water and of dry air are to each other as 8 to 5; and the increase of height in barometric formulæ due to the consideration of the hygrometric state of the atmosphere requires such a calculation as that employed in this paper.

Although the error belonging to the formulæ given in the "Proceedings of the Royal Irish Academy," vol. II. p. 565, and vol. III. p. 322, be not considerable, and although they are an obvious improvement upon the formulæ in previous use, yet as they are erroneous in principle, they ought not to be retained, seeing that they are replaced by the more correct formulæ of the present paper.

N. B.—The equation already published (*vide* "Proceedings of the Royal Irish Academy," vol. II. p. 565, and vol. III. p. 322), viz.:

$$v' = v \times \frac{P}{P-F}, \text{ in which } \frac{\frac{1}{M} \cdot \log \frac{p}{p'}}{\frac{1}{p} - \frac{1}{p'}} = P, \text{ and } \sqrt{(ff')} = F,$$

in order to harmonize with this paper, ought to be

$$v' = v \times \frac{P}{P - \frac{3}{8}F}.$$